# CHAPTER 1

## Review of basic Quantum Mechanics concepts

*Introduction. Hermitian operators. Physical meaning of the eigenvectors and eigenvalues of Hermitian operators. Representations and their use. Non-Hermitian and Unitary Operator: symmetries and conservation laws. Sum of angular momenta. 3j, 6j and 9j symbols.*

### Introduction

In Quantum Mechanics the states are represented by vectors in an abstract space called Hilbert space. Thus, a state $\alpha$ is a vector which, in Dirac notation, is writen as $|\alpha>$. As we will see below, this vector can be associated either to a function $\Psi_\alpha(\vec{r})$, which is regular and square integrable, or to a one-dimensional matrix (spinor). In the first case the metric of the space is defined by the scalar product in the region $V$ of the three-dimensional (physical) space where the functions are square integrable. Usually $V$ includes the whole space. Thus, the scalar product between the vectors $|\alpha>$ and $|\beta>$ is defined by

$$\langle\beta|\alpha\rangle = \int_V d\vec{r}\ \Psi^*_\beta(\vec{r})\Psi_\alpha(\vec{r})$$

For the case of an N-dimensional spinor the vector $\alpha$ is associated to the one-dimensional matrix given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_N \end{pmatrix}$$

and the scalar product between the vectors $\alpha$ and $\beta$ is given by

$$\langle\beta|\alpha\rangle = \begin{pmatrix} \beta^*_1, & \beta^*_2, & ., & ., & ., & \beta^*_N \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_N \end{pmatrix} = \sum_{i=1}^{N} \beta^*_i \alpha_i$$

The vector $<\alpha|$ is called "bra" and $|\alpha>$ is called "ket". The scalar produc $<\alpha|\beta>$ is called "bracket".

One sees from the definition of the scalar product that it is $<\alpha|\beta>=<\beta|\alpha>^*$. Therefore the norm $N_\alpha$ of a vector $|\alpha>$, i. e. $N_\alpha = \sqrt{<\alpha|\alpha>}$ is a real number. In Quantum Mechanics $N^2_\alpha$ is the probability of measuring the system in the state $\alpha$. Since the system exists, this probability should be $N^2_\alpha = 1$. Notice that we assume that the system is stationary, that is all processes are time-independent. Therefore if the system is in the state $\alpha$, it will remain there for ever. Below we will describe this system in terms of a set of vectors $|n>$. The probability of measuring the system in the state $|n>$ is $|<\alpha|n>|^2$.

**Hermitian operators**

An operator $A$ acting upon a vector $|\alpha\rangle$ in the Hilbert space converts this vector into another one $|\beta\rangle$. It is important to point out that the Hilbert space we consider is closed, that is all vectors belong to the space. In the applications that we will encounter in the course of these lectures only small subspaces of the total Hilbert space (which usually has infinite dimension) will be chosen. In such a case the operator $\hat{A}$ may bring $|\alpha>$ to a vector $|\beta>$ lying outside the subspace. But we will not treat such situations here. In other words, the systems we will treat are always closed.

The relation between $|\alpha>$ and $|\beta>$ is $\hat{A}|\alpha\rangle = |\beta\rangle$. The corresponding adjoint operator $\hat{A}^+$ is defined by $\langle\beta| = \langle\alpha|\hat{A}^\dagger$. Taking the scalar product with another vector $\gamma$ one gets, $\langle\gamma|\hat{A}|\alpha\rangle = \langle\gamma|\beta\rangle = \langle\beta|\gamma\rangle^* = \langle\alpha|\hat{A}^\dagger|\gamma\rangle^*$. The operator $\hat{A}$ is called Hermitian if

$$\hat{A}^\dagger = \hat{A} \qquad \text{Hermitian}$$

The eigenvalues $a$ and eigenvectors $|\alpha>$ of the operator $\hat{A}$ satisfy the equation

$$\hat{A}|\alpha\rangle = a|\alpha\rangle$$

and for the adjoint operator $A^+$ it is,

$$\langle\alpha|\hat{A}^\dagger = a^*\langle\alpha|$$

If the operator $\hat{A}$ is Hermitian one has

$$\langle\beta|\hat{A}|\alpha\rangle = \langle\beta|\hat{A}^\dagger|\alpha\rangle$$

If, in addition, $|\alpha>$ and $|\beta>$ are eigenvectors of $\hat{A}$, then one gets

$$a\langle\beta|\alpha\rangle = b^*\langle\beta|\alpha\rangle \tag{1}$$

Which implies that

$$\begin{cases} |\alpha\rangle \neq |\beta\rangle; & \langle\beta|\alpha\rangle = 0 \\ |\alpha\rangle = |\beta\rangle; & a \text{ real} \end{cases} \tag{2}$$

That is, the eigenvectors of a Hermitian operator form an orthogonal set and the corresponding eigenvalues are real. These two properties play a fundamental role in Quantum Mechanics, as we repeatly will see in the course of these lectures.

**Physical meaning of the eigenvectors and eigenvalues of Hermitian operators**

The most important property of Hermitian operators in Quantum Mechanics is that their eigenvalues are real. This property has allowed one to interpret these operators as the devices used to measure physical quantities. One postulates that the Hermitian operator represents the apparatus used to measure a physical quantity and the corresponding eigenvalues are all the possible values that one can obtain from the measurement. In other words, only those values are allowed and nothing else. This is a radical departure from Classical Mechanics, where one can give any value one wishes to all physical quantities (for instance the energy).

The eigenvectors of Hermitian operators are the corresponding wave functions that allow one to evaluate all probabilities, in particular transition probabilities. Besides, they play a fundamental role in Quantum Mechanics. Thus, normalizing the eigenvectors in Eq. (1) as $\langle\alpha|\alpha\rangle = 1$ and from Eq. (2) one finds that they satisfy

$$\langle\alpha|\beta\rangle = \delta_{\alpha\beta} \tag{3}$$

which means that they form an orthonormal set of vectors in the Hilbert space. They can be used as a basis to describe any vector belonging to the space. In a more rigorous statement one can say that the eigenvectors of an Hermitian operator spann the Hilbert space on which the operator acts. To see the great importance of this property, assume a Hilbert space of dimension $N$ and a Hermitian operator $\hat{A}$ acting on this space such that,

$$\hat{A}|\alpha_i\rangle = a_i|\alpha_i\rangle, \qquad i = 1, 2, \cdots, N$$

Any vector $|v\rangle$ in the space spanned by the basis $\{|\alpha\rangle\}$ can be written as

$$|v\rangle = \sum_{i=1}^{N} c_i|\alpha_i\rangle$$

From Eq. (3) one obtains

$$c_i = \langle\alpha_i|v\rangle \tag{4}$$

The numbers $\langle\alpha_i|v\rangle$ are called "amplitudes". If the vector $|v>$ represents a physical (quantum) state, then the amplitudes have to obey the normality relation given by,

$$\langle v|v\rangle = \sum_{i=1}^{N} c_i\langle v|\alpha_i\rangle = \sum_{i=1}^{N}\langle v|\alpha_i\rangle^*\langle v|\alpha_i\rangle = \sum_{i=1}^{N}|\langle v|\alpha_i\rangle|^2 = 1$$

From Eq. (4) the vector $|v\rangle$ can be written as,

$$|v\rangle = \sum_{i=1}^{N}\langle\alpha_i|v\rangle|\alpha_i\rangle = \sum_{i=1}^{N}|\alpha_i\rangle\langle\alpha_i|v\rangle$$

which shows that

$$\sum_{i=1}^{N}|\alpha_i\rangle\langle\alpha_i| = \hat{I} \tag{5}$$

This is the projector into the space spanned by the set $\{|\alpha\rangle\}$. We will use the projector often in these lectures.

**Representations and their use**

One of the most important problems in theoretical nuclear physics is to evaluate the eigenvectors and eigenvalues of a given operator $\hat{B}$. That is, to find the vectors $\beta$ and numbers $b$ defined by

$$\hat{B}|\beta\rangle = b|\beta\rangle$$

This is called the "Eigenvalue problem". To evaluate the eigenvectors and eigenvalues one first chooses a basis, that is a set of orthonormal vectors $\{|\alpha\rangle\}$ which are usually provided by the diagonalization of a Hermitian operator. This basis is also called "Representation", for reason which will become clear below. If the number of vectors forming the orthonormal set $\{|\alpha\rangle\}$, i. e. the dimension of the basis, is $N$ one has, applying Eq. (5)

$$\hat{B}|\beta\rangle = b|\beta\rangle = \hat{B}\sum_{i=1}^{N}|\alpha_i\rangle\langle\alpha_i|\beta\rangle \Longrightarrow \sum_{i=1}^{N}\left(\langle\alpha_j|\hat{B}|\alpha_i\rangle - b\delta_{ij}\right)\langle\alpha_i|\beta\rangle = 0$$

which provides the equation to evaluate the amplitudes as,

$$\sum_{i=1}^{N}\left(\langle\alpha_j|\hat{B}|\alpha_i\rangle - b\delta_{ij}\right)\langle\alpha_i|\beta\rangle = 0 \qquad (6)$$

This is a set of $N \times N$ homogeneous linear equations in the $N$ unknowns amplitudes $\langle\alpha_i|\beta\rangle$. Besides the trivial solution $\langle\alpha_i|\beta\rangle$=0 for all $i$, one finds the physical solutions by requiring that the equations (6) are linearly dependent upon each other. This occurs if the corresponding determinant vanishes. That is

$$\left|\left|\langle\alpha_j|\hat{B}|\alpha_i\rangle - b\delta_{ij}\right|\right| = 0$$

which allows one to calculate $N$ values of $b$. To calculate the amplitudes one disregard one of the Eqs. (6) and the remaining $N-1$ equations plus the normalization condition given by

$$\sum_{i=1}^{N}|\langle\alpha_i|\beta\rangle|^2 = 1$$

give a non-linear $N \times N$ set of equations from which the amplitudes $\langle\alpha_i|\beta\rangle$ are extracted.

The set $\{|\alpha\rangle\}$ can be a continuum set. An example of this is the eigenvectors corresponding to the distance operator, i. e.

$$\hat{r}|\boldsymbol{r}\rangle = r|\boldsymbol{r}\rangle$$

The operator $\hat{r}$ represents the device used to measure the distance (a rule for instance), $|\boldsymbol{r}\rangle$ is the corresponding vector in the Hilbert space and $r$ the length one measures. Since

$$\hat{r} = \hat{r}^{\dagger}$$

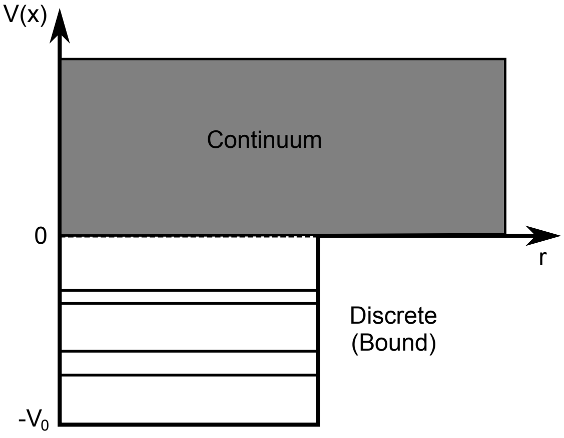


Figure 1: Potential that induces both discrete (bound) and continuum states.

one gets the projector as

$$\int \mathrm{d}\boldsymbol{r}|\boldsymbol{r}\rangle\langle\boldsymbol{r}| = \hat{I} \tag{7}$$

where, in spherical coordinates, it is $\boldsymbol{r} = (r, \theta, \varphi)$ and $\mathrm{d}\boldsymbol{r} = r^2\mathrm{d}r \sin\theta \mathrm{d}\theta\mathrm{d}\varphi$. One cannot speak of a number of dimensions of this continuous basis, since it comprises all real numbers (which cannot be labelled by integers). Therefeore one uses the name "representation" for the projector (7). In $\boldsymbol{r}$-representation the eigenvalue problem is

$$\hat{B}|\beta\rangle = b|\beta\rangle = \hat{B}\int \mathrm{d}\boldsymbol{r}'|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'|\beta\rangle$$

$$\hat{B}\int \mathrm{d}\boldsymbol{r}'\langle\boldsymbol{r}|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'|\beta\rangle = b\langle\boldsymbol{r}|\beta\rangle$$

With $\langle\boldsymbol{r}|\boldsymbol{r}'\rangle = \delta(\boldsymbol{r} - \boldsymbol{r}')$ and $\Psi_\beta(\boldsymbol{r}) = \langle\boldsymbol{r}|\beta\rangle$, one gets

$$\hat{B}\Psi_\beta(\boldsymbol{r}) = b\Psi_\beta(\boldsymbol{r})$$

and the matrix elements can readily be evaluated as,

$$\langle\alpha_j|\hat{B}|\alpha_i\rangle = \int \mathrm{d}\boldsymbol{r}\int \mathrm{d}\boldsymbol{r}'\langle\alpha_j|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\hat{B}|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'|\alpha_i\rangle = \int \mathrm{d}\boldsymbol{r}\Psi_j^*(\boldsymbol{r})\hat{B}\Psi_i(\boldsymbol{r})$$

There can be discrete and continuum states in some cases, as seen in Fig. 1. In these cases the projector becomes,

$$\hat{I} = \sum_{n=1}^{N}|n\rangle\langle n| + \int \mathrm{d}\alpha|\alpha\rangle\langle\alpha|$$

where $|n>$ ($|\alpha>$) is a discrete (continuum) state.

The orthonormalization condition now reads,

$$\delta(\boldsymbol{r}-\boldsymbol{r}') = \sum_{n=1}^{N} \Psi_n^*(\boldsymbol{r})\Psi_n(\boldsymbol{r}') + \int \mathrm{d}\alpha \Psi_\alpha^*(\boldsymbol{r})\Psi_\alpha(\boldsymbol{r}')$$

**Non-Hermitian and Unitary Operator: symmetries and conservation laws**

Operators are of a fundamental importance to describe transformations of the system. We will analyse in this Lecture the cases of translations, rotations and parity.

*1.- Translation symmetry*

The translation operator $\hat{T}$ is defined by,

$$\hat{T}(\Delta\boldsymbol{r})|\boldsymbol{r}\rangle = |\boldsymbol{r}+\Delta\boldsymbol{r}\rangle$$

which applied to the vector $|\Psi\rangle$ gives,

$$\hat{T}(\Delta\boldsymbol{r})|\Psi\rangle = \int \mathrm{d}\boldsymbol{r}'\hat{T}(\Delta\boldsymbol{r})|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'|\Psi\rangle = \int \mathrm{d}\boldsymbol{r}'|\boldsymbol{r}'+\Delta\boldsymbol{r}\rangle\langle\boldsymbol{r}'|\Psi\rangle$$

and, in r-representation, the translated function is

$$\Psi_t(\boldsymbol{r}) = \langle\boldsymbol{r}|\hat{T}(\Delta\boldsymbol{r})|\Psi\rangle = \int \mathrm{d}\boldsymbol{r}'\delta(\boldsymbol{r}-\boldsymbol{r}'-\Delta\boldsymbol{r})\langle\boldsymbol{r}'|\Psi\rangle = \Psi(\boldsymbol{r}-\Delta\boldsymbol{r}) \tag{8}$$

Since

$$\langle\boldsymbol{r}|\hat{T}^\dagger(\Delta\boldsymbol{r})\hat{T}(\Delta\boldsymbol{r})|\boldsymbol{r}\rangle = \langle\boldsymbol{r}+\Delta\boldsymbol{r}|\boldsymbol{r}+\Delta\boldsymbol{r}\rangle = 1$$

one obtains

$$\hat{T}^\dagger\hat{T} = 1$$

which defines the operator $\hat{T}$ as unitary.

The invariance of a wave function with respect to translations implies the conservation of the linear momentum, This can be seen by noticing that the time dependence of an operator $\hat{A}$ is given by

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \frac{\partial\hat{A}}{\partial t} + \frac{\mathrm{i}}{\hbar}[H,\hat{A}] \tag{9}$$

Assume a system for which

$$H\Psi_n(x) = E_n\Psi_n(x)$$

If there is translation invariance, then

$$H\Psi_n(x+\Delta x) = E_n\Psi_n(x+\Delta x)$$

Making a Taylor expansion of $\Psi_n$ one gets,

$$\Psi_n(x+\Delta x) = \sum_{k=1}^{\infty} \frac{(\Delta x)^k}{k\,!} \frac{\mathrm{d}^k \Psi_n(x)}{\mathrm{d}x^k} = \sum_{k=1}^{\infty} \frac{1}{k\,!} \left( \Delta x \frac{\mathrm{d}}{\mathrm{d}x} \right)^k \Psi_n(x) = \mathrm{e}^{\Delta x \frac{\mathrm{d}}{\mathrm{d}x}} \Psi_n(x)$$

and defining the linear momentum operator in the usual fashion as,

$$p_x = \frac{\hbar}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x}$$

one obtains,

$$\Psi_n(x+\Delta x) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Delta x p_x} \Psi_n(x)$$

Therefore the translation operator is

$$\hat{T}(\Delta x) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Delta x p_x}$$

and one has

$$\begin{aligned} H\hat{T}(\Delta x)\Psi_n(x) &= H\Psi_n(x+\Delta x) \\ &= E_n \Psi_n(x+\Delta x) \\ &= E_n \hat{T}(\Delta x)\Psi_n(x) = \hat{T}(\Delta x) H \Psi_n(x) \end{aligned}$$

which implies,

$$[H, \hat{T}] = 0$$

i. e.

$$[H, p_x] = 0$$

Since $p_x$ is time independent it is $\partial \hat{p_x}/\partial dt$ =0 and from Eq. (9) one gets $\mathrm{d}\hat{p_x}/\mathrm{d}t$=0, which means that the linear momentum is conserved.

*2.- Rotational symmetry*

Performing a rotation of the system by an angle $\delta\boldsymbol{\varphi}$, as shown in Fig. 2, a function $\Psi(\boldsymbol{r})$ is transformed to $\Psi(\boldsymbol{r}+\boldsymbol{a})$. As seen in the Figure, it is $a = \delta\varphi\ r sin\theta$ and the relation among the vectors $\boldsymbol{a}$, $\delta\boldsymbol{\varphi}$ and $\boldsymbol{r}$ is

$$\boldsymbol{a} = \delta\boldsymbol{\varphi} \times \boldsymbol{r}$$

Calling

$$F(\boldsymbol{r}) = \Psi(\boldsymbol{r}+\boldsymbol{a})$$

one gets

$$\Psi(\boldsymbol{r}) = F(\boldsymbol{r}-\boldsymbol{a}) = F(\boldsymbol{r}) - \boldsymbol{a}\cdot\boldsymbol{\nabla}F(\boldsymbol{r}) + \cdots$$

Performing a Taylor expansion as,

$$F(x+\Delta x, y+\Delta y, z+\Delta z) = \\ F(x,y,z) + \frac{\partial F(x,y,z)}{\partial x}\Delta x + \frac{\partial F(x,y,z)}{\partial y}\Delta y + \frac{\partial F(x,y,z)}{\partial z}\Delta z + \cdots \quad (10)$$

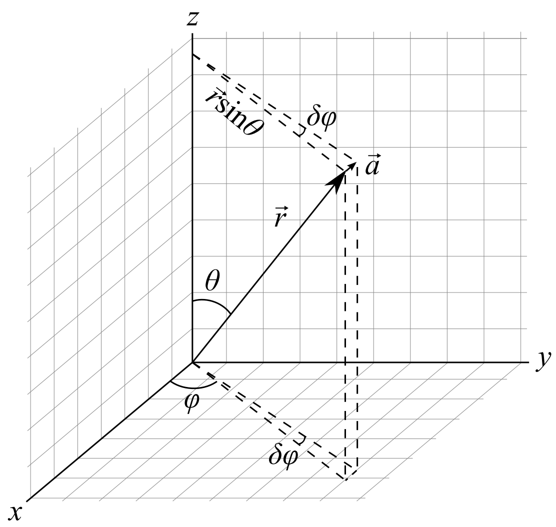


Figure 2: Angle $\delta\varphi$ corresponding to the rotation of the system and the relation among the radius $\vec{r}$, the radius increment $\vec{a}$ and the angle increment $\delta\varphi$.

one gets,

$$F(\boldsymbol{r}+\Delta\boldsymbol{r}) = F(\boldsymbol{r}) + \Delta\boldsymbol{r}\cdot\boldsymbol{\nabla}F(\boldsymbol{r}) + \cdots$$

With $\delta\boldsymbol{\varphi}$ infinitesimal it is,

$$\Psi(\boldsymbol{r}) - F(\boldsymbol{r}) = -(\delta\boldsymbol{\varphi}\times\boldsymbol{r})\cdot\boldsymbol{\nabla}F(\boldsymbol{r}) = -\delta\boldsymbol{\varphi}\cdot(\boldsymbol{r}\times\boldsymbol{\nabla}F(\boldsymbol{r}))$$

and replacing momenta by the corresponding operators one gets,

$$\boldsymbol{p} = \frac{\hbar}{\mathrm{i}}\boldsymbol{\nabla} \Longrightarrow \boldsymbol{r}\times\boldsymbol{\nabla} = \frac{\mathrm{i}}{\hbar}\boldsymbol{r}\times\boldsymbol{p} = \frac{\mathrm{i}}{\hbar}\boldsymbol{L}$$

Finally one obtains,

$$\Psi(\boldsymbol{r}) = F(\boldsymbol{r}) - \frac{\mathrm{i}}{\hbar}\delta\boldsymbol{\varphi}\cdot\boldsymbol{L}F(\boldsymbol{r}) = \left(1 - \frac{\mathrm{i}}{\hbar}\delta\boldsymbol{\varphi}\cdot\boldsymbol{L}\right)F(\boldsymbol{r})$$

For a finite angle $\boldsymbol{\varphi}$ one defines a small angle by using $\delta\boldsymbol{\varphi} = \boldsymbol{\varphi}/n$, where $n$ is a large number. Rotating $n$ times, i. e. applying the rotation operator $n$ times, and in the limit of $n = \infty$, one gets,

$$\Psi(\boldsymbol{r}) = \lim_{n\to\infty}\left(1 - \frac{\mathrm{i}}{\hbar}\frac{\boldsymbol{\varphi}\cdot\boldsymbol{L}}{n}\right)^n F(\boldsymbol{r})$$

Since the Euler's number e is

$$\mathrm{e} = \lim_{n\to\infty}(1+1/n)^n$$

one can write

$$\Psi(\boldsymbol{r}) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\boldsymbol{\varphi}\cdot\boldsymbol{L}}F(\boldsymbol{r}) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\boldsymbol{\varphi}\cdot\boldsymbol{L}}\Psi(\boldsymbol{r}+\boldsymbol{a})$$

Therefore the rotation operator is

$$U_R = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\boldsymbol{\varphi}\cdot\boldsymbol{L}}$$

and one gets

$$\Psi(\boldsymbol{r}+\boldsymbol{a}) = U_R^{-1}\Psi(\boldsymbol{r})$$

If the Hamiltonian is rotational invariant

$$H\Psi_n(\boldsymbol{r}+\boldsymbol{a}) = E_n\Psi(\boldsymbol{r}+\boldsymbol{a}) = E_n U_R^{-1}\Psi(\boldsymbol{r}) = U_R^{-1}H\Psi(\boldsymbol{r})$$

$$U_R H\Psi_n(\boldsymbol{r}+\boldsymbol{a}) = U_R H U_R^{-1}\Psi(\boldsymbol{r}) = H\Psi(\boldsymbol{r})$$

$$[H, U_R] = 0 \Longrightarrow [H, \boldsymbol{L}] = 0$$

That is, if there is rotational invariance, then the angular momentum is conserved.

*3.- Parity symmetry*

The parity operator $\hat{\pi}$ is defined as,

$$\hat{\pi}|x\rangle = |-x\rangle$$

it is $\hat{\pi}^\dagger = \hat{\pi}$ (exercise) The eigenvalues of the parity operator are obtained as,

$$\hat{\pi}|\Psi_\lambda\rangle = \lambda|\Psi_\lambda\rangle \Longrightarrow \hat{\pi}^2|\Psi_\lambda\rangle = \lambda^2|\Psi_\lambda\rangle$$

since

$$\hat{\pi}^2|x\rangle = |x\rangle$$

one gets

$$\lambda^2|\Psi_\lambda\rangle = |\Psi_\lambda\rangle \Longrightarrow \lambda = \pm 1$$

in $x$-space it is

$$\langle x|\hat{\pi}|\Psi_\lambda\rangle = \langle -x|\Psi_\lambda\rangle = \Psi_\lambda(-x) = \lambda\Psi_\lambda(x)$$

$$\Psi_\lambda(x) = \begin{cases} \text{even}, & \lambda = 1 \\ \text{odd}, & \lambda = -1 \end{cases}$$

If $[H, \hat{\pi}]$=0, as it happens with potentials with reflection symmetry, parity is conserved and $\lambda$ is a good quantum number.

### Sum of angular momenta

We will here analyse the possible angular momenta values of a two-particle system. The angular momenta of the particles are $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ and the total angular momentum is $\boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2$. The components $\hat{L}_x, \hat{L}_y, \hat{L}_z$ of $\boldsymbol{L}$ satisfy the conmutation relations

$$[\hat{\boldsymbol{L}}^2, \hat{L}_i] = 0 \quad (i = x, y, z)$$

$$[\hat{L}_x, \hat{L}_y] = \mathrm{i}\hbar\hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = \mathrm{i}\hbar\hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = \mathrm{i}\hbar\hat{L}_y$$

and the same for $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$.

Besides, since the degrees of freedom of the particles are independent of each other one also has,

$$[\hat{\boldsymbol{L}}_1, \hat{\boldsymbol{L}}_2] = 0, \quad [\hat{\boldsymbol{L}}^2, \hat{\boldsymbol{L}}_1] = [\hat{\boldsymbol{L}}^2, \hat{\boldsymbol{L}}_2] = 0$$

The eigenvectors corresponding to these operators are given by

$$\begin{array}{rclcrcl} \hat{\boldsymbol{L}}_1^2|l_1m_1\rangle &=& \hbar^2 l_1(l_1+1)|l_1m_1\rangle &;& \hat{L}_{1z}|l_1m_1\rangle &=& \hbar m_1|l_1m_1\rangle \\ \hat{\boldsymbol{L}}_2^2|l_2m_2\rangle &=& \hbar^2 l_2(l_2+1)|l_2m_2\rangle &;& \hat{L}_{2z}|l_2m_2\rangle &=& \hbar m_2|l_2m_2\rangle \\ \hat{\boldsymbol{L}}^2|lm\rangle &=& \hbar^2 l(l+1)|lm\rangle &;& \hat{L}_z|lm\rangle &=& \hbar m|lm\rangle \end{array}$$

$$|l_1 - l_2| \leqslant l \leqslant l_1 + l_2, \quad m = m_1 + m_2$$
$$-l_i \leqslant m_i \leqslant l_i, \quad -l \leqslant m \leqslant l$$

Not all the quantum numbers related to these operators can be used to label the states. In other words, not all of them can be taken as good quantum numbers. To see the reason for this we will analyse the behaviour of conmuting operators.

Given two operators and their eigenstates as

$$\hat{A}|\alpha\rangle = a|\alpha\rangle \quad \text{and} \quad \hat{B}|\beta\rangle = b|\beta\rangle$$

and assuming that they conmute, i. e. $[\hat{A}, \hat{B}] = 0$, then they have common eigenvalues (see Homeworproblems 1), i. e.,

$$\hat{A}|\alpha\beta\rangle = a|\alpha\beta\rangle, \quad \hat{B}|\alpha\beta\rangle = b|\alpha\beta\rangle$$

Therefore one cannot choose as quantum numbers to label simultaneously the states the eigenvalues of, e. g., $\hat{\boldsymbol{L}}_1^2$, $\hat{L}_{1z}$, $\hat{L}_{2x}$ and $\hat{L}_{2z}$, since these two last operators do not conmute with each other. But there are many combinations one can choose. For instance, one can choose the eigenvalues of $\hat{\boldsymbol{L}}_1^2$, $\hat{L}_{1x}$, $\hat{\boldsymbol{L}}_2^2$, $\hat{L}_{2x}$. However, it is standard in Quantum Mechanics to choose as quantum numbers the eigenvalues of the $z$-component of all angular momenta. Therefore the standard choose (which corresponds to all existing Tables of angular momentum coefficients) is $\hat{\boldsymbol{L}}_1^2$, $\hat{L}_{1z}$, $\hat{\boldsymbol{L}}_2^2$, $\hat{L}_{2z}$ or $\hat{\boldsymbol{L}}_1^2$, $\hat{\boldsymbol{L}}_2^2$, $\hat{\boldsymbol{L}}^2$, $\hat{L}_z$, i.e. the standard eigenvectors used to label the angular momenta are

$$|l_1m_1l_2m_2\rangle \quad \text{or} \quad |l_1l_2lm\rangle$$

and, therefore, the standard projectors are

$$\sum_{l_1m_1l_2m_2} |l_1m_1l_2m_2\rangle\langle l_1m_1l_2m_2| = \hat{I} \quad \text{or} \quad \sum_{l_1l_2lm} |l_1l_2lm\rangle\langle l_1l_2lm| = \hat{I}$$

One can write the vector in one representation in terms of the other representation, for instance

$$|l_1m_1l_2m_2\rangle = \sum_{lm} |l_1l_2lm\rangle\langle l_1l_2lm|l_1m_1l_2m_2\rangle$$

The number $\langle l_1m_1l_2m_2|lm\rangle = \langle l_1l_2lm|l_1m_1l_2m_2\rangle$ is real and is called Clebsch-Gordan coefficient.

$$|l_1m_1l_2m_2\rangle = \sum_{lm} \langle l_1m_1l_2m_2|lm\rangle |l_1l_2lm\rangle \tag{11}$$

and due to the orthonormality of the basis elements

$$|l_1 l_2 l m\rangle = \sum_{m_1 m_2} \langle l_1 m_1 l_2 m_2 | l m\rangle | l_1 m_1 l_2 m_2\rangle \tag{12}$$

If the Hamiltonian corresponding to the two-particle system is spherically symmetric then the eigenstates of the Hamiltonian can be labeled by the eigenvalues of the angular momenta shown above.

**Symmetry properties of the Clebsch-Gordan coefficient**

The Clebsch-Gordan coefficient can best be written in terms of the 3-j symbol defined as

$$\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & -m \end{pmatrix} = \frac{(-1)^{l_1-l_2+m}}{\sqrt{2l+1}} \langle l_1 m_1 l_2 m_2 | l m\rangle$$

with the properties that

1. $\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} = \begin{pmatrix} l_2 & l & l_1 \\ m_2 & m & m_1 \end{pmatrix} = \begin{pmatrix} l & l_1 & l_2 \\ m & m_1 & m_2 \end{pmatrix}$
2. $\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{l_1+l_2+l} \begin{pmatrix} l_2 & l_1 & l \\ m_2 & m_1 & m \end{pmatrix}$
3. $\begin{pmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m \end{pmatrix} = (-1)^{l_1+l_2+l} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix}$
4. $m_1 + m_2 - m = 0$

**6-j symbols**

In the sum of three angular momenta one can choose the partition

$$\boldsymbol{J} = \boldsymbol{j}_1 + \boldsymbol{j}_2 + \boldsymbol{j}_3 = \boldsymbol{J}_{12} + \boldsymbol{j}_3 = \boldsymbol{j}_1 + \boldsymbol{J}_{23}$$

where

$$\boldsymbol{J}_{12} = \boldsymbol{j}_1 + \boldsymbol{j}_2, \quad \boldsymbol{J}_{23} = \boldsymbol{j}_2 + \boldsymbol{j}_3$$

One can write the basis vector in one representation in terms of the other representation as

$$|(j_1 j_2) J_{12} j_3; JM\rangle = \sum_{J_{23}} \langle j_1 (j_2 j_3) J_{23}; J | (j_1 j_2) J_{12} j_3; J\rangle | j_1 (j_2 j_3) J_{23}; JM\rangle$$

The symmetry properties of the expansion coefficient can best be seen by introducing the 6-j symbol as

$$\begin{aligned} &\langle j_1 (j_2 j_3) J_{23}; J | (j_1 j_2) J_{12} j_3; J\rangle \\ &\qquad = (-1)^{j_1+j_2+j_3+J} \sqrt{(2J_{12}+1)(2J_{23}+1)} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix} \end{aligned} \tag{13}$$

which is a real number (therefore it is the same for $\langle (j_1 j_2) J_{12} j_3; J | j_1 (j_2 j_3) J_{23}; J \rangle$).
The 6-j symbol does not change if two columns are inter changed, for instance

$$\left\{ \begin{array}{ccc} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{array} \right\} = \left\{ \begin{array}{ccc} j_1 & J_{12} & j_2 \\ j_3 & J_{23} & J \end{array} \right\}$$

The angular momentum triangular relation must be satisfied for $(j_1, j_2, J_{12})$, $(j_1, J, J_{23})$, $(j_3, j_2, J_{23})$ and $(j_3, J, J_{12})$. Thus, e.g.,

$$\left\{ \begin{array}{ccc} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 2 \end{array} \right\} = 0$$

since $1/2 + 1/2 < 2$

**9-j symbols**

In the case of 4 angular momenta

$$\boldsymbol{J} = \boldsymbol{j}_1 + \boldsymbol{j}_2 + \boldsymbol{j}_3 + \boldsymbol{j}_4$$

one can write, e.g.,

$$\boldsymbol{J} = \boldsymbol{J}_{12} + \boldsymbol{J}_{34} = \boldsymbol{J}_{13} + \boldsymbol{J}_{24}$$

where $\boldsymbol{J}_{12} = \boldsymbol{j}_1 + \boldsymbol{j}_2$, $\boldsymbol{J}_{34} = \boldsymbol{j}_3 + \boldsymbol{j}_4$, $\boldsymbol{J}_{13} = \boldsymbol{j}_1 + \boldsymbol{j}_3$ and $\boldsymbol{J}_{24} = \boldsymbol{j}_2 + \boldsymbol{j}_4$.

One can thus write

$$\begin{aligned} |(j_1 j_3) J_{13} (j_2 j_4) J_{24}; JM\rangle &= \sum_{J_{12} J_{34}} \langle (j_1 j_2) J_{12} (j_3 j_4) J_{34}; J | (j_1 j_3) J_{13} (j_2 j_4) J_{24}; JM \rangle \\ &\quad \times |(j_1 j_2) J_{12} (j_3 j_4) J_{34}; J\rangle \end{aligned} \tag{14}$$

and the 9-j symbol is defined by,

$$\begin{aligned} &\langle (j_1 j_2) J_{12} (j_3 j_4) J_{34}; J | (j_1 j_3) J_{13} (j_2 j_4) J_{24}; J \rangle \\ &\qquad = \sqrt{(2J_{12}+1)(2J_{34}+1)(2J_{13}+1)(2J_{24}+1)} \left\{ \begin{array}{ccc} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{array} \right\} \end{aligned} \tag{15}$$

which is also a real number.

The symmetry properties of the 9-j symbols are

1. Any permutation of rows and columns does not change the 9-j symbol except the sign, which is plus if the permutation is even and $(-1)^S$, where $S$ is the sum of all angular momenta, if the permutation is odd.

2. The 9-j symbol does not change under a reflection about either diagonal.